Advances
in Applied Mathematics

An Introduction to Partial Differential Equations with MATLAB®

3rd edition

Chapter 1: Introduction

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1.1 What are Partial Differential Equations?

Given a function $u = u(x_1, x_2, \ldots, x_n)$, a **partial differential equation** in u is an equation that relates any of the partial derivatives of u to each other and/or any of the variables $x_1, x_2, \ldots, x_n$, and u.

Notation:

$$u_x = \frac{\partial u}{\partial x}, \qquad u_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x},$$

also

$$u_{xzyx} = u_{zxxy} = u_{yxzx},$$

etc.

The **order** of a PDE is the order of the highest derivative which appears in the equation.

Example 1: $u_{xxz}^5 + u_{xxyz} = u_{zzz}$ is a PDE of 4th order in $u(x, y, z)$.

A **solution** of a PDE is any function u which satisfies the PDE identically, that is, for all possible values of the independent variables.

Example 2: $u(x, y) = cx + d$ is a solution of

$$u_{xxz}^5 + u_{xxyz} = u_{zzz}$$

for any choice of the constants c and d.

1.1 What are Partial Differential Equations? (cont'd)

Required: any solution u of an nth-order PDE has the property that all of the nth partial derivatives of u exist and are continuous.

Some important PDEs:

$u_t + cu_x = 0$	**convection** (also **advection** or **transport**) **equation**
$u_t + uu_x = 0$	**Burger's equation** (from the study of the dynamics of gases)
$u_x^2 + u_y^2 = 1$	**eikonal equation** (from optics)
$u_t = \alpha^2 u_{xx}$	**heat equation** (in one space variable)
$u_{tt} = c^2 (u_{xx} + u_{yy})$	**wave equation** (in two space variables)
$u_{xx} + u_{yy} + u_{zz} + [E - V(x, y, z)]u = 0$	**Schrödinger's equation** (time independent, in three space variables; from quantum mechanics)
$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$	**Laplace's equation** (in polar coordinates)
$u_{tt} + \alpha^4 u_{xxxx} = 0$	**Euler–Bernoulli beam equation**

1.2 PDEs We Can Already Solve

Method: "antidifferentiate" with respect to one variable while treating the other variables as constants.

Example 1: Find all functions $u = u(x, y)$ which solve

$$u_x = 0.$$

We get

$$u = \int 0 \, dx = f(y)$$

where f is any arbitrary function of y (and where $\int \dots dx$ is any antiderivative with respect to x while treating y as a constant).

Since

$$u = f(y)$$

represents all possible solutions of $u_x = 0$, we call it the **general solution** of $u_x = 0$.

So, where the general solution of an ODE involves arbitrary constants, the general solution of a PDE involves **arbitrary functions**.

1.2 PDEs We Can Already Solve (cont'd)

Example 2: Do the same for $u_{xy} = \cos x$. First, we have

$$u_x = \int \cos x \, dy = y \cos x + f(x).$$

Then,

$$u = \int (y \cos x + f(x)) \, dx.$$

Now, what is $\int f(x)\,dx$? If we antidifferentiate $f(x)$ with respect to x, we just get another function of x. However, we also get an "arbitrary constant," that is, in this case, an arbitrary function of y. So

$$\int f(x)\,dx = f_1(x) + g(y),$$

and our general solution is

$$u = y \sin x + f_1(x) + g(y).$$

Finally, since f, f_1, and g are arbitrary, we drop the subscript, i.e.,

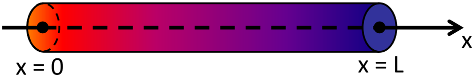
$$u = y \sin x + f(x) + g(y),$$

where f and g are arbitrary functions.

1.3 Initial and Boundary Conditions

Heat equation:

$$u_t = \alpha^2 u_{xx},$$



e.g., for metal rod, where $u(x, t) =$ temperature at point x, at time t.

Initial condition: initial temperature of the material at each point x, at some specified time $t = t_0$ given by function f

$$u(x, t_0) = f(x), \qquad 0 \leq x \leq L.$$

Boundary conditions: we need to know what is going on at the endpoints. For example, the left end may be held at a constant temperature u_0

$$u(0, t) = u_0, \qquad t > 0,$$

and the right end may be insulated, i.e.,

$$u_x(L, t) = 0, \qquad t > 0.$$

This is an example of **initial-boundary-value problem** (IBVP).

Compare: IBVP and **initial-value problem** (IVP).

1.3 Initial and Boundary Conditions (cont'd)

Well-posed problem: the initial-boundary-value problem has a unique solution.

Precise definition: an initial-value or initial-boundary-value problem is well-posed if

1. A solution to it exists.
2. There is only one such solution (i.e., the solution is unique).
3. The problem is stable (no concerns in this book!).

To determine a unique solution for ODEs of order n: we generally need n initial conditions.

For PDEs, the situation is much more complicated! Notice that heat equation has one time derivative and one initial condition, while it has two x-derivatives and two x-boundary conditions. It is often the case!

1.4 Linear PDEs – Definitions

Linear PDEs are defined in exactly the same manner as linear ODEs:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x),$$

or using the so-called **linear operator** L

$$L[y] = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x),$$

such that

$$L[cy] = cL[y] \qquad \text{and} \qquad L[y_1 + y_2] = L[y_1] + L[y_2]$$

for any constant c and any functions y_1 and y_2 in the domain of L.

Definition: *The PDE in $u = u(x_1, x_2, \ldots, x_n)$*

$$L[u] = f(x_1, x_2, \ldots, x_n)$$

is a **linear PDE** *if*

$$L[cu] = cL[u]$$

for all constants c and all functions u in the domain of L, and

$$L[u_1 + u_2] = L[u_1] + L[u_2],$$

for all functions u_1 and u_2 in the domain of L. Also, if an operator satisfies both conditions, we say that it is a **linear operator** *(otherwise, it is a* **nonlinear operator***).*

1.4 Linear PDEs – Definitions (cont'd)

We can prove that L is linear if and only if

$$L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2]$$

for all constants c_1 and c_2 and all functions u_1 and u_2 in the domain of L.

Example: $y^2 u_{xx} + u_{yy} = 1$. Here, $L[u] = y^2 u_{xx} + u_{yy}$ and

$$\begin{aligned} L[c_1 u_1 + c_2 u_2] &= y^2 (c_1 u_1 + c_2 u_2)_{xx} + (c_1 u_1 + c_2 u_2)_{yy} \\ &= c_1 y^2 u_{1_{xx}} + c_2 y^2 u_{2_{xx}} + c_1 u_{1_{yy}} + c_2 u_{2_{yy}} \\ &= c_1 \left(y^2 u_{1_{xx}} + u_{1_{yy}} \right) + c_2 \left(y^2 u_{2_{xx}} + u_{2_{yy}} \right) \\ &= c_1 L[u_1] + c_2 L[u_2], \end{aligned}$$

so this PDE is linear.

Definition: *Given the linear PDE $L[u] = f$, if $f \equiv 0$ on some region (that is, f is the zero-function on some region), we say that the PDE is* **homogeneous** *on that region. Otherwise, the PDE is* **nonhomogeneous***.*

Example 1. The PDE $xu_{xx} - 5u_{xy} + y^2 u_x = 0$ is homogeneous (on the x-y plane).

Example 2. $u_x^2 + u_y^2 = 0$ cannot be said to be homogeneous or nonhomogeneous because it is not a linear PDE to start with.

1.5 Linear PDEs – The Principle of Superposition

Definition: *Given functions $u_1, u_2, \ldots, u_n$, any function of the form*

$$c_1 u_1 + c_2 u_2 + \ldots + c_n u_n,$$

where $c_1, c_2, \ldots, c_n$ are constants, is called a **linear combination** *of $u_1, u_2, \ldots, u_n$.*

Theorem (**principle of superposition of solutions** for linear PDEs): *If $u_1, u_2, \ldots, u_n$ are solutions of the linear, homogeneous PDE $L[u] = 0$, then so is any linear combination of $u_1, u_2, \ldots, u_n$.*

Proof:

The fact that $u_1, u_2, \ldots, u_n$ are solutions gives

$$L[u_1] = L[u_2] = \ldots = L[u_n] = 0.$$

Then, for any linear combination $c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$,

$$\begin{aligned} L[c_1 u_1 + c_2 u_2 + \ldots + c_n u_n] &= c_1 L[u_1] + c_2 L[u_2] + \ldots + c_n L[u_n] \\ &= c_1 \cdot 0 + c_2 \cdot 0 + \ldots + c_n \cdot 0 = 0. \end{aligned}$$

1.5 Linear PDEs – The Principle of Superposition (cont'd)

Recall: for a linear homogeneous ODE-n, we need only find n linearly independent solutions, and its general solution consists of all possible (finite) linear combinations of these solutions.

However, life is much more complicated in the realm of PDEs! Often, we will need to find infinitely many solutions, $u_1, u_2, \ldots$, of a linear homogeneous PDE to construct its general solution

$$u = c_1 u_1 + c_2 u_2 + \ldots = \sum_{n=1}^{\infty} c_n u_n. \tag{1}$$

Question of **convergence** – this infinite linear combination is an infinite series – for any given choice of the coefficients, (1) may diverge for all values of x, or it may converge for some values of x but not for others.

Assumption: whenever (1) converges, it satisfies the linearity condition

$$L\left[\sum_{n=1}^{\infty} c_n u_n\right] = \sum_{n=1}^{\infty} c_n L\left[u_n\right]. \tag{2}$$

Therefore, if each u_n is a solution of $L[u] = 0$, then so is the linear combination (1) of these solutions. When (2) holds, we say that we may differentiate the series term-by-term.

1.6 The Method of Characteristics I

Example 1: Find all solutions of the PDE

$$2u_x + 3u_y = 0.$$

We apply transformation

$$\xi = x, \qquad \eta = Ax + By$$

and choose constants A and B so that the transformed PDE has no u_η term:

$$u_x = u_\xi + Au_\eta, \qquad u_y = Bu_\eta.$$

The PDE becomes

$$2u_\xi + (2A + 3B)u_\eta = 0.$$

Choosing $A = 3$ and $B = -2$

$$\begin{aligned} \xi &= x \\ \eta &= 3x - 2y \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= \xi \\ y &= \frac{3}{2}\xi - \frac{1}{2}\eta \end{aligned}$$

reduces PDE to

$$2u_\xi = 0,$$

where

$$u = g(\eta) = g(3x - 2y)$$

and g is any arbitrary function (must be differentiable).

1.6 The Method of Characteristics I (cont'd)

Notice: for all points on the line $3x - 2y = c$, where c is a given constant, we have

$$u(x, y) = g(3x - 2y) = g(c),$$

that is, u is constant along each of the lines $3x - 2y = c$. These important lines are called the **characteristics** or **characteristic curves** of the PDE, while ξ and η are called **characteristic coordinates** (definition of characteristics comes later in this chapter.)

Example 1 (cont'd): Suppose that the PDE is to be solved subject to the additional condition

$$u(x, 0) = \sin x.$$

Then,

$$u(x, 0) = g(3x) = \sin x$$

and, letting $z = 3x$, $x = \frac{1}{3}z$, we have $g(z) = \sin\left(\frac{1}{3}z\right)$. The unique solution to the system

$$2u_x + 3u_y = 0,$$
$$u(x, 0) = \sin x$$

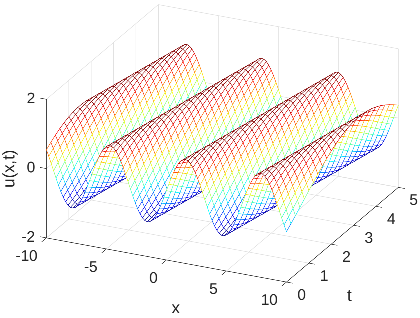
is

$$u(x, y) = \sin \frac{1}{3}(3x - 2y) = \sin\left(x - \frac{2}{3}y\right).$$

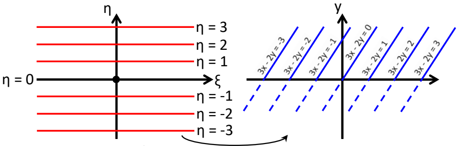
1.6 The Method of Characteristics I (cont'd)

The condition $u(x, 0) = \sin x$ is called a **side condition** (**initial condition** if $y = t$). The curve along which the condition is given is called the **initial curve**, and the system is an **initial-value problem**.

Relationship between the solution in the transformed coordinates (ξ, η) and the actual solution:



M: `Chapter_1_characteristics.m`



(a) characteristics in (ξ,η)-plane (b) characteristics in (x,y)-plane

Graphical interpretation: We replace y by the time variable, t. If we take "snapshots" of the solution $u(x, t) = \sin\left(x - \frac{2}{3}t\right)$ in the (x, u)-plane at various times t_0, we can think of our solution as an initial curve $u = \sin x$, which moves to the right at constant velocity.

More generally: given the first-order linear PDE

$$au_x + bu_y + cu = f(x, y),$$

where a, b, and c are constant, we may always proceed as above. We find that the transformation

$$\begin{aligned} \xi &= x \\ \eta &= bx - ay \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= \xi \\ y &= \frac{b}{a}\xi - \frac{1}{a}\eta \end{aligned}$$

(for example) reduces this PDE to

$$au_\xi + cu = F(\xi, \eta),$$

where $F(\xi, \eta)$ is just the function $f(x, y)$ with $x = \xi$ and $y = \frac{b}{a}\xi - \frac{1}{a}\eta$.

Example 2: Solve

$$u_x - 4u_y + u = 0,$$
$$u(0, y) = \cos 3y.$$

Again, we let

$$\xi = x \qquad \text{and} \qquad \eta = Ax + By$$

and the transformed PDE becomes

$$u_\xi + (A - 4B)u_\eta + u = 0.$$

Example 2 (cont'd): We may choose $A = 4$ and $B = 1$, so that

$$\xi = x \qquad \text{and} \qquad \eta = 4x + y \qquad \Rightarrow \qquad u_\xi + u = 0.$$

Its solution is

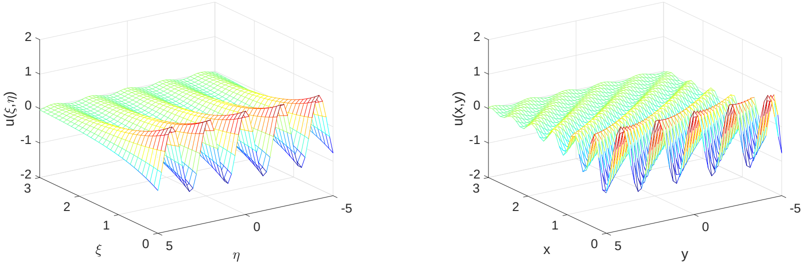
$$u = g(\eta)e^{-\xi} \qquad \text{or} \qquad u = g(4x + y)e^{-x},$$

where g is any function. Then, applying the initial condition, we have

$$u(0, y) = g(y) = \cos 3y,$$

so our final, unique solution is

$$u(x, y) = e^{-x} \cos 3(4x + y).$$



Visualization in MATLAB: `Chapter_1_characteristics.m`

1.6 The Method of Characteristics I (cont'd)

What's going on is this:

- The PDE becomes a first-order ODE along each of the characteristic curves.
- In order to have a unique solution, the initial condition must specify the value of u at exactly one point on each of these characteristics.
- Therefore, it seems that the curve along which the initial condition u is given must intersect each characteristic at exactly one point.

Theorem: *Given the initial-value problem*

$$au_x + bu_y + cu = f(x, y),$$
$$u(x, f_1(x)) = f_2(x),$$

where a, b, and c are constant, suppose that

1) *f_x, f_y, f_1', and f_2' are continuous.*
2) *Each characteristic of the PDE intersects the initial curve $y = f_1(x)$ exactly once.*
3) *No characteristic is tangent to the initial curve.*

Then, the initial-value problem has a unique solution u, with the property that u_x and u_y are continuous.

1.7 The Method of Characteristics II

Can we extend the method of characteristics to deal with equations where the coefficients are not all constant?

Example 1: Find all solutions of $u_x + 4xu_y - u = 0$. The term $u_x + 4xu_y$ is the directional derivative of u in the direction of the vector $\hat{i} + 4x\hat{j}$. We expect curves with this vector as tangent vectors to be the characteristics that satisfy $\frac{dy}{dx} = \frac{4x}{1}$, so the characteristics are the curves

$$y = 2x^2 + c \qquad \text{or} \qquad 2x^2 - y = constant.$$

Now, let

$$\begin{aligned} \xi &= x \\ \eta &= 2x^2 - y \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= \xi \\ y &= 2\xi^2 - \eta. \end{aligned}$$

Then

$$u_x = u_\xi + 4xu_\eta \qquad \text{and} \qquad u_y = -u_\eta$$

and the transformed PDE is

$$u_\xi - u = 0$$

with solution

$$u = g(\eta)e^\xi = g(2x^2 - y)e^x$$

for arbitrary function g. So the curves $2x^2 - y = c$ are, indeed, characteristic.

1.7 The Method of Characteristics II (cont'd)

More precisely, given the first-order linear PDE

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y),$$

we define the characteristic curves to be those curves satisfying

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

or, as is traditionally written,

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)}.$$

Definition: *The* **characteristics** *or* **characteristic curves** *of the first-order linear PDE*

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$

are those curves satisfying the ODE

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)}.$$

1.7 The Method of Characteristics II (cont'd)

Supposing that the ODE $\dfrac{dx}{a(x,y)} = \dfrac{dy}{b(x,y)}$ has general solution $h(x,y) = c$, we make the transformation

$$\begin{aligned} \xi &= x \\ \eta &= h(x,y). \end{aligned}$$

In this case, we have

$$\begin{aligned} u_x &= u_\xi + u_\eta h_x \\ u_y &= u_\eta h_y \end{aligned}$$

and

$$au_x + bu_y = au_\xi + (ah_x + bh_y)u_\eta.$$

But

$$\frac{dx}{a} = \frac{dy}{b} \quad \Rightarrow \quad h(x,y) = c$$

$$\begin{aligned} \Rightarrow \quad dh = 0 = h_x dx + h_y dy &= dx\left(h_x + h_y \frac{dy}{dx}\right) \\ &= dx\left(h_x + h_y \frac{b}{a}\right) = \frac{dx}{a}(ah_x + bh_y) \end{aligned}$$

and we have

$$au_x + bu_y = au_\xi,$$

so the PDE has been reduced to an ODE.

1.7 The Method of Characteristics II (cont'd)

Example 2: Solve

$$u_x + yu_y = x,$$
$$u(1,y) = \cos y.$$

The characteristics are given by $\dfrac{dx}{1} = \dfrac{dy}{y}$ with solution

$$y = ce^{x} \qquad \text{or} \qquad ye^{-x} = c.$$

Then, our transformation is

$$\begin{aligned} \xi &= x \\ \eta &= ye^{-x} \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= \xi \\ y &= \eta e^{\xi} \end{aligned}$$

and the PDE becomes $u_\xi = \xi$ with solution

$$u = \frac{\xi^2}{2} + g(\eta) = \frac{x^2}{2} + g(ye^{-x}).$$

Finally,

$$u(1,y) = \cos y = \frac{1}{2} + g(ye^{-1}),$$

and, letting $z = \frac{y}{e}$ and $y = ez$, we get $g(z) = \cos ez - \dfrac{1}{2}$, and the unique solution is

$$u = \frac{x^2}{2} + \cos(ye^{1-x}) - \frac{1}{2}.$$

1.8 Separation of Variables for Linear, Homogeneous PDEs

Definition: *Given a PDE in $u = u(x, y)$, we say that u is a* **product solution** *if*

$$u(x, y) = f(x)g(y)$$

for functions f and g. More generally, $u = u(x_1, x_2, \dots, x_n)$ is a product solution of a PDE in the n variables $x_1, x_2, \dots, x_n$ if

$$u(x_1, x_2, \dots, x_n) = f_1(x_1)f_2(x_2)\dots f_n(x_n)$$

for functions $f_1, f_2, \dots, f_n$.

Example 1: Find all product solutions of the first-order, linear, homogeneous PDE

$$u_x + u_y = 0.$$

We search for all solutions of the form $u(x, y) = X(x)Y(y)$:

$$u_x = X'Y \quad \text{and} \quad u_y = XY',$$

$$X'Y + XY' = 0,$$

$$\frac{X'}{X} = -\frac{Y'}{Y}.$$

We have managed to **separate** the variable x from the variable y. We say that the equation is **separable** and that we have **separated the variables**.

1.8 Separation of Variables for Linear, Homogeneous PDEs (cont'd)

Example 1 (cont'd): We have a situation

$$f(x) = g(y)$$

for all values of x and y in the domain of the problem. We have

$$u(x, y) = X(x)Y(y) \text{ as a solution to } \frac{X'}{X} = -\frac{Y'}{Y} = \lambda$$

for some real constant λ. This equation actually is two equations:

$$\frac{X'}{X} = \lambda \quad \text{and} \quad \frac{Y'}{Y} = -\lambda,$$

$$X' - \lambda X = 0 \quad \text{and} \quad Y' + \lambda Y = 0.$$

The product solutions are

$$X(x) = e^{\lambda x} \quad \text{and} \quad Y(y) = e^{-\lambda y}$$

or

$$u(x, y) = e^{\lambda(x-y)}$$

for any real constant λ. Further, any linear combination of these solutions is a solution.

Note: we shall find only certain values of λ that will lead to nontrivial (other than the zero-function) solutions of the problem.

Example 2: Find all product solutions of the heat equation $u_t = u_{xx}$.

$$u(x,t) = X(x)T(t) \qquad \Rightarrow \qquad X(x)T'(t) = X''(x)T(t),$$

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda,$$

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + \lambda T = 0.$$

We must consider three cases:

Case 1: $\lambda > 0 \quad \Rightarrow \quad X = c\cos\sqrt{\lambda}x + d\sin\sqrt{\lambda}x, \quad T = e^{-\lambda t}$

$$u = e^{-\lambda t}\left[c\cos\sqrt{\lambda}x + d\sin\sqrt{\lambda}x\right]$$

Case 2: $\lambda = 0 \quad \Rightarrow \quad X = cx + d, \quad T = 1, \quad u = cx + d$

Case 3: $\lambda < 0 \quad \Rightarrow \quad X = ce^{\sqrt{-\lambda}x} + de^{-\sqrt{-\lambda}x}, \quad T = e^{-\lambda t}$

$$u = e^{-\lambda t}\left[ce^{\sqrt{-\lambda}x} + de^{-\sqrt{-\lambda}x}\right]$$

In each case, c and d are arbitrary constants.

Again, any linear combination of solutions is a solution.

1.9 Eigenvalue Problems

Note:

- We solve the X-ODE for each real number λ with infinitely many solutions.
- We need to find which of these solutions "survive" the boundary conditions.
- For "most" real numbers λ, the only solution that also satisfies the boundary conditions is the zero-solution, $X(x) \equiv 0$.
- Thus, we need to identify those values of λ for which the X-system has nontrivial solutions.

These values of λ are called **eigenvalues** of the X-system, and the corresponding nontrivial solutions are the **eigenfunctions** associated with λ. The system itself is an example of an ODE **eigenvalue problem**.

Example 1: Find all eigenvalues and eigenfunctions of the eigenvalue problem

$$y'' + \lambda y = 0,$$
$$y'(0) = y'(3) = 0.$$

Case 1: $\lambda < 0$, $\lambda = -k^2$, $k > 0$

$$y = c_1 \cosh kx + c_2 \sinh kx, \quad \Rightarrow \quad y' = c_1 k \sinh kx + c_2 k \cosh kx,$$
$$y'(0) = c_2 k = 0 \;\Rightarrow\; c_2 = 0, \qquad y'(3) = c_1 k \sinh 3k = 0 \;\Rightarrow\; c_1 = 0,$$

so there are no negative eigenvalues.

1.9 Eigenvalue Problems (cont'd)

Case 2: $\lambda = 0$

$$y = c_1 x + c_2 \quad \Rightarrow \quad y' = c_1,$$
$$y'(0) = y'(3) = c_1 = 0.$$

Here, $y = c_2$ survives both boundary conditions, so $\lambda_0 = 0$ is an eigenvalue with eigenfunction $y_0 = 1$.

Case 3: $\lambda > 0$, $\lambda = k^2$, $k > 0$

$$y = c_1 \cos kx + c_2 \sin kx \quad \Rightarrow \quad y' = -c_1 k \sin kx + c_2 k \cos kx,$$
$$y'(0) = c_2 k = 0 \;\Rightarrow\; c_2 = 0,$$
$$y'(3) = -c_1 k \sin 3k = 0 \;\Rightarrow\; c_1 = 0$$

unless

$$\sin 3k = 0, \text{ that is, } 3k = \pi, 2\pi, 3\pi, \ldots \quad \text{or} \quad k = \frac{n\pi}{3}, \quad n = 1, 2, 3, \ldots .$$

Therefore, we have eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{9}, \quad n = 1, 2, 3, \ldots$$

with associated eigenfunctions

$$y_n = \cos \frac{n\pi x}{3}, \quad n = 1, 2, 3, \ldots .$$

1.9 Eigenvalue Problems (cont'd)

Example 2: Do the same for

$$y'' + \lambda y = 0,$$
$$y(0) = y(1) + y'(1) = 0.$$

Case 1: $\lambda < 0$, $\lambda = -k^2$, $k > 0$

The problem has no negative eigenvalues.

Case 2: $\lambda = 0$ is not an eigenvalue.

Case 3: $\lambda > 0$, $\lambda = k^2$, $k > 0$

$$y = c_1 \cos kx + c_2 \sin kx \quad \Rightarrow \quad y' = -c_1 k \sin kx + c_2 k \cos kx.$$

Then,

$$y(0) = c_1 = 0$$
$$y(1) + y'(1) = c_2(\sin k + k \cos k).$$

This system has only the solution $c_1 = c_2 = 0$ unless k is such that

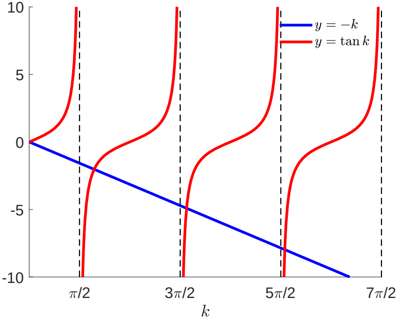
$$\sin k + k \cos k = 0.$$

Therefore, the eigenvalues correspond to those values of k satisfying

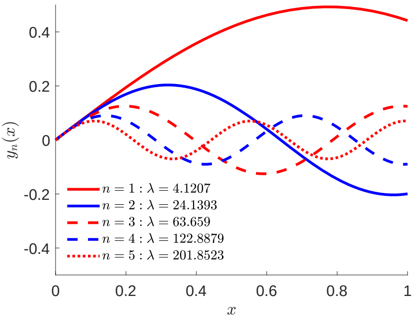
$$-k = \tan k.$$

1.9 Eigenvalue Problems (cont'd)

How do we solve for k? We don't – because we can't! However, we can show that there are infinitely many such values of k for $k > 0$:



M: `Chapter_1_graph_intersection.m`



M: `Chapter_1_bvp4c_eigenproblem.m`

Therefore, the eigenvalues are those $\lambda_n > 0$ satisfying $-\sqrt{\lambda_n} = \tan\sqrt{\lambda_n}$, with associated eigenfunctions $y_n = \sin\left(\sqrt{\lambda_n}x\right)$.

MATLAB's routine `bvp4c`: refer to Chapter 3 and `Chapter_1_bvp4c_eigenproblem.m`